[redacted]

BRL R 273

BRL

AD 704864

REPORT NO. 273

THE PRECISE DETERMINATION OF THE POSITION OF A POINT IN SPACE, FROM PHOTOGRAPHS TAKEN AT TWO GROUND STATIONS

by

T. E. Sterne

April 1942

U.S. ARMY ABERDEEN RESEARCH AND DEVELOPMENT CENTER
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

Ballistic Research
Laboratory Report No. 273

TES/abh
Aberdeen Proving Ground, Md.,
April 13, 1942.

THE PRECISE DETERMINATION OF THE POSITION OF A POINT IN SPACE, FROM PHOTOGRAPHS TAKEN AT TWO GROUND STATIONS

Abstract

Simultaneous photographs are taken, of the same aerial point, by two cameras on the ground. Stars also are photographed, on some or all of the plates. Here the mathematical procedure is developed and described for finding accurately the position of the aerial point with respect to a set of terrestrial axes, ξ , η , and ζ . The formulae are designed to expedite the numerical calculations as much as possible, consistently with an accuracy of the order of a second of arc; i.e., of better than a foot at 50,000 feet. Another less accurate procedure is also described that does not involve star-images on all of the plates. The present discussion is applicable to the case where both camera axes are vertical. Some of the formulae are applicable also to oblique camera axes.

1. Introduction. In order to obtain the fundamental data from which bomb ballistic tables can be prepared, it is necessary to know the position and velocity of an airplane at the instant when it releases a bomb. Both the position and velocity can be determined from a knowledge of the position of the airplane at each of several known times and from a knowledge of the time of release. Nearly instantaneous light signals can be emitted from the airplane at known times, and each of these signals can be photographed by two cameras mounted rigidly on terrestrial piers. The problem with which we are here concerned is the determination of the position in space of each of these light signals, from the information provided by such photographs.

The position of an aerial point is determined when its directions from both cameras are known. In order to be of use in range bombings from great altitudes, these directions must be found to a very high accuracy. The most accurate, and yet not prohibitively laborious or lengthy procedure is to photograph known "comparison" stars on the same plates that photograph the aerial points. Then, by adapting to the present problem the simple, mechanical, and yet exceedingly accurate methods of photographic astrometry, one can find the directions of the rays to the aerial points from a knowledge of the positions of the comparison stars, accurately listed in star catalogues. The x and y coordinates, of the images of the stars and of the aerial points on the photographic plates, must be measured on a measuring engine. The star-images determine a set of "plate constants", and the plate constants in turn determine the direction ratios to the aerial points. It is not necessary for the ways of the engine to be exactly perpendicular to each other or parallel to the edges of the plate; it is not necessary for the scales of the measuring engine, in x and y, to be the same; no fiducial marks are necessary on the plates; the focal lengths of the cameras need not be accurately known; and errors in the assumed orientations of the cameras produce only second-order errors in the resulting directions to the aerial points.

An alternative and less accurate procedure is to determine the plate constants on some plates from the star-images, and to find the directions of the rays to an aerial point on other plates by employing fiducial marks impressed upon the plates by the camera. The alternative procedure obviates the need for employing star-images on all the plates, but presumes a very high degree of fixity and permanence in the mountings and orientations of the cameras.

Here we develop and present the necessary formulae, many of which are mere adaptations of the well-known formulae of photographic astrometry. Astrometry, however, concerns the relative directions of stars with respect to each other; in the present problem we have also to find directions relative to the surface of the earth. The present problem, therefore, has some new aspects. Further, it is desirable to simplify some of the conventional astrometric formulae with a view to accelerating routine computation, to diminishing the frequency of errors of computation, and to facilitating the checking of the various stages of the reductions. To avoid errors arising from the entering of tables incorrectly, an effort will be made to avoid the use of trigonometric functions, by using direction cosines instead. After some preliminary transformations have been made to direction cosines, no further use will be made of trigonometric functions. It is believed that the new forms into which some of the old astrometric formulae have been recast may be useful outside of the present problem in astrometry.

The use of two terrestrial cameras was urged by Professor Henry Norris Russell, to whom the writer is grateful also for his valuable and specific suggestions. Among them were the finding of lens-distortion by examining the residuals, of a least squares solution for plate constants. He also pointed out that whenever the "base" of a plate is the zenith, then the direction cosines in the altitude-azimuth system, to any point, can be found at once from its standard coordinates.

We take up the problem in the order in which the computations should be made. We discuss first the reduction of mean star positions, found from a star-catalogue, to apparent positions. Here the most expeditious procedure, since standard coordinates must be computed, is to compute direction cosines at once and then correct the direction cosines, rather than the right ascensions and declinations, for precession since the beginning of the year, for nutation, and for aberration. Then standard coordinates are to be computed, to be corrected for lens-distortion and third-order terms in refraction. Then the plate constants are found, and from them and the plate-coordinates of the aerial point one computes the direction ratios of the rays to the aerial point. From these (corrected for refraction and distortion) the terrestrial coordinates of the aerial point follow at once.

In subsequent sections we show how to follow the alternative procedure that does not use star-places on all the plates.

Equations to be used by computers are numbered with Roman, other equations with Arabic, numbers.

2. <u>The Reduction of the Star-Places</u>. Reduce the catalogue mean positions of the stars to the beginning of the current year, by applying the annual and secular variations (which include proper motion) listed in the catalogue. Apply also the proper motion, if it amounts to more than $0''.1$ (or $0^s.01$) from the beginning of the year to date, and apply further the small diurnal aberration. The equations are

$$\alpha = \alpha_0 + \dot{\alpha}t + \frac{\ddot{\alpha}t^2}{200} + \mu t' + 0^s.02$$

$$\delta = \delta_0 + \dot{\delta}t + \frac{\ddot{\delta}t^2}{200} + \mu' t' \qquad \text{(I)}$$

where

μ, μ'	are the annual proper motions in right ascension and declination,
α, δ	are the right ascension and declination, for the mean equinox of the beginning of the current year, and epoch of date, affected by zenithal diurnal aberration,
α_0, δ_0	are the catalogue right ascension and declination (for the mean equinox of the catalogue)
$\dot{\alpha}, \dot{\delta}$	are the catalogue annual variations,
$\ddot{\alpha}, \ddot{\delta}$	are the secular variations in right ascension and declination,
$\underline{t}$	is the number of whole years elapsed since the epoch of the catalogue,
$\underline{t}'$	is the fraction of the current year elapsed, and,
	$0^s.02$ is the zenithal diurnal aberration.

NOTE: The symbols t and μ will be used later in different senses.

It will be noted that $\underline{t}$ and $\underline{t}^2/200$, as well as $\underline{t}'$, are the same for all the stars used and may be computed once and for all for any date. The above reductions are, therefore, very rapidly performed if $\underline{t}$, $\underline{t}^2/200$, and $\underline{t}'$ be entered on a slip of paper. If the Boss General Catalogue is used, $\underline{t}$ for 1942 is, of course, -8, and $\underline{t}^2/200$ is +0.32. The terms in $\underline{t}'$ are usually ignorable.

A word of explanation is required about the diurnal aberration. Its effect upon the apparent position is

$$\begin{aligned} d\alpha &= 0''.31 \cos\varphi \cos T \sec\delta \\ d\delta &= 0''.31 \cos\varphi \sin T \sin\delta \end{aligned} \tag{2}$$

where T is the hour-angle and φ the latitude. It can thus never exceed 0".31 in absolute amount, in either right ascension or in declination. With vertical cameras the stars photographed will be close to the zenith -- with the Goerz cameras, within 20° of it. At the zenith

$$d\alpha = 0^s.021, \quad d\delta = 0.$$

It follows from equations (2) that at Aberdeen Proving Ground, the diurnal aberration 20° from the zenith can differ from its zenithal value by no more than 0".16. Thus the equations (I) allow correctly for diurnal aberration to within this high accuracy.

Compute equatorial direction cosines, L_o, M_o, N_o, for each star by the formulae

$$\begin{aligned} L_o &= \cos\delta \cos\alpha \\ M_o &= \cos\delta \sin\alpha \\ N_o &= \sin\delta . \end{aligned} \tag{III}$$

It will be noticed that these cosines relate to the mean equator and equinox of the beginning of the current year, and contain the diurnal aberration, and the proper motion since the beginning of the year. These cosines will thus never need to be computed more than once in each year, except, rarely, in the case of large proper motions.

It is next necessary to correct these cosines for precession since the beginning of the year, for nutation, and for annual aberration. This correction is most readily carried out by the formulae

$$L = L_o + \Delta L$$
$$M = M_o + \Delta M \qquad \text{(IV)}$$
$$N = N_o + \Delta N$$

where the corrections ΔL, ΔM, and ΔN are given by the formulae

$$P = DL_o - CM_o - iN_o$$
$$10{,}000\ \Delta L = PL_o - FM_o - AN_o - D$$
$$10{,}000\ \Delta M = FL_o + PM_o + BN_o + C \qquad \text{(V)}$$
$$10{,}000\ \Delta N = AL_o - BM_o + PN_o + i$$

In these equations the coefficients A, B, C, D, F, and i depend only on the date, and should be tabulated to an accuracy of 0.001 for convenient reference in the use of the present formulae. They are defined by

$$B = 0.04848\ B_e$$
$$C = 0.04848\ C_e$$
$$D = 0.04848\ D_e$$
$$F = 0.72722\ (f_e + f'_e)$$
$$i = 0.04848\ i_e$$
$$A = 0.9717\ A_e \text{ from 1937 through 1960, and,}$$
$$A = 0.9716\ A_e \text{ from 1961 through 1985,}$$

where the symbols A_e, B_e, C_e, D_e, f_e f'_e and i_e denote the quantities that are listed in the American Ephemeris and Nautical Almanac, for each day in the year, in the section dealing with star reductions, under the names "A", "B", "C", "D", "f", "f'", and "i" without the subscript "e". These quantities are all in seconds of arc with the exception of A_e, a pure number, and f_e and f'_e which are in seconds of time. A, B, C, D, in the notation of the Almanac are Besselian star numbers; f, f', and i are Independent star numbers.

The equations (IV) and (V) have been derived merely by writing down the first derivatives of L, M, and N with respect to α and δ, and employing the formulae for $d\alpha$ and $d\delta$ that are listed on page 222 of the American Ephemeris and Nautical Almanac for 1938 under the title "Independent Star Numbers". The terms in the proper motion can be dropped from the latter formulae since proper motion has been already allowed for. Formulae (IV) and (V) follow after some algebraic reductions, which the reader can readily verify. The second-order terms in ΔL, ΔM, and ΔN are of the order of only 0.0000001, or 0".02, and have been here ignored.

In using equations (V), it is sufficient to know the coefficients only to the nearest 0.001 and to use approximate values of L_o, M_o, and N_o rounded off to the nearest 0.001. Then ΔL, ΔM, and ΔN will be accurate to the nearest 0.0000001, which is sufficient. It is a good plan to retain seven digits in L_o, M_o, N_o, L, M, and N and throughout the subsequent stages, except where otherwise specified in following sections. We have already pointed out that the cosines L_o, M_o, N_o can be computed once and for all for a whole year of observation except in the rare cases of stars of large proper motion. Likewise, L, M, N will stay effectively constant on any one night and will vary only slightly from one night to the next.

The computation of L, M, N can be checked by verifying that the sum of their squares is unity, and this check should previously have been applied to L_o, M_o, and N_o.

3. Standard Coordinates. Denote by τ the sidereal time of the photograph of the star-images. This time will differ for the two cameras even if they are exposed simultaneously, unless they lie on the same meridian of longitude; denote by $\underline{l}$, $\underline{m}$, $\underline{n}$ the direction cosines of a star referred to the axes of the hour-angle declination system. Denote the ordinary equatorial system of axes by x", y", z" with the x" towards the equinox, and z" towards the north pole of the sky. Denote the hour-angle declination system of axes by x', y', z', with x' pointing towards the intersection of the equator with the observer's meridian, and with the z' axis continuing to point to the north pole. The primed system of axes is obtained by rotating the unprimed system positively through the angle τ, and thus

$$l = L \cos\tau + M \sin\tau$$

$$m = L \sin\tau + M \cos\tau \qquad \text{(VI)}$$

$$n = N$$

The factors cos τ and sin τ must be looked up, and then all the stars that one is using must have their old cosines, L, M, and N transformed by the equations (VI) into the new cosines l, m, n. The new cosines may be checked by verifying that the sum of squares is unity, and are thus rapidly computed and verified.

It is important (if one is using the *American Ephemeris and Nautical Almanac* for computing the sidereal time from the standard time) that the *true* sidereal time, including terms of short period in the nutation, must be used and *not* the uniform sidereal time. The "sidereal time of 0^h" listed in the *Almanac* *may* be used properly for the conversion; but *not* the "civil time of sidereal 0^h", in which short-period terms have been ignored.

By the *base* of a photographic plate, we mean the image on the plate of the infinitely distant point whose rays enter the camera perpendicularly to the plate. With a horizontal plate, the base is the image of the camera's zenith, of which the hour-angle is zero and of which the declination is equal to the latitude, φ, of the camera. We are here concerned with a horizontal plate, whose base is the zenith; in the next section we shall discuss the magnitude of the errors in direction that can result from errors in levelling, the former errors being much smaller than the latter.

Consider a set of axes OX, OY, OZ with OX parallel to the y' axis, OZ towards the camera's zenith, and OY in the plane ZOz' and, therefore, in the terrestrial north direction. Consider a plane Z = K where K is some constant. The X and Y coordinates of the intersection of this plane (called the "standard coordinate plane") with a ray OQ are said to be the "standard coordinates" corresponding to the direction OQ. Let the direction cosines of OQ in the x', y', z' system be l, m, n. The cosines of the angles between the axes of the two systems of coordinates are

	X	Y	Z
x'	0	$-\sin\varphi$	$\cos\varphi$
y'	1	0	0
z'	0	$\cos\varphi$	$\sin\varphi$

Thus the direction cosines of OQ in the X, Y, Z system are

$$m, \qquad -l \sin\varphi + n\cos\varphi, \qquad l\cos\varphi + n\sin\varphi$$

and the intersections desired are Z = 1, and,

$$X = K\frac{m}{\beta l + \gamma n}$$

$$Y = K\frac{\beta n - \gamma l}{\beta l + \gamma n} \tag{VII}$$

where $\beta = \cos\varphi$, $\gamma = \sin\varphi$. The factors β and γ are mere constants dependent only on the latitude of the camera, and thus the equations (VII) allow the standard coordinates, X and Y, of any star to be readily computed. For the Goerz cameras the greatest angular departure from the OZ axis is about 20°, and it is, therefore, suggested that the constant K be set equal to 10 in order not to have too small X's and Y's. Equation (VII') may be used to check the computation of X and Y:

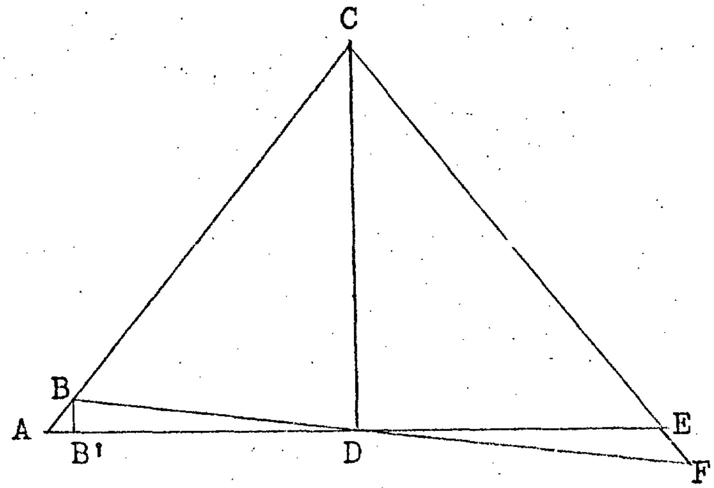
$$X^2 + Y^2 + K^2 \equiv K^2/(\beta l + \gamma n)^2. \tag{VII'}$$

The equations (VII) may also be derived from those of Turner* or Schlesinger**, who employ a different notation, if one transforms to direction cosines after inserting in their formulae a base whose declination equals the latitude, and whose right ascension is equal to that of the zenith.

* Turner, Monthly Notices Royal Astronomical Society, V54, 11, 1893.

** Schlesinger, Transactions of the Yale University Observatory, V4 No.10.

4. Accuracy of Levelling the Plate. The standard coordinates will be used, in a process that is really a type of interpolation, to find the directions of images on the plate from a knowledge of the directions of known star-images. It is easy to find approximately the effect of small errors of levelling upon the accuracy of directions thus interpolated. In the accompanying diagram, C is the principal point of the camera lens, CD is vertical, AE is the assumed horizontal position of the plate, BF is its actual position, A and E are two star-images such that AD = DE. Denote the error of levelling, namely the angle BDA, by ε ; denote the angle ACD by ζ . Then A and E



correspond to the standard coordinates computed for an assumed level plate. The distance from B to B' is nearly ε (AD) where B' is the foot of the perpendicular from B to AE; and thus AB' is nearly ε (AD) ζ . Now D is the mid-point of AE, and differs from the mid-point of BF by the distance AB'. If D is compared with the images B and F on the tilted plate, one will, therefore, infer, ignoring the tilt, a direction for CD that is in error by the angle (AB')/(CD), or $\varepsilon\zeta^2$, nearly.

Angular errors can therefore arise, in the directions of measured images inferred from the star-images, equal to the angular error of the base of the plate multiplied by the square of the angular semi-field of the plate. The semi-field for the Goerz camera is about 20°, or one-third of a radian; consequently, the greatest error in an inferred direction will be about one-ninth of the angular error of the adopted base. If the zenith is adopted as the base, then an error of less than 1" of arc will result from a departure of the plate of 9" from horizontal. This limit of 9" can readily be met by careful levelling, causing the greatest errors arising from uncertainty of the base to be smaller than one second of arc.

5. Corrections for Lens Distortion and Astronomical Refraction. The camera lens distorts, causing images to be displaced radially away from or towards the base of the plate. For any but a very poor lens, or one badly tilted in its cell, the distortion is purely radial; and the effect of atmospheric refraction is likewise (with the plates "based" at the zenith) purely radial from the base. Both effects can therefore be considered at once, since they are similar. Further, it is easier to consider effects and corrections to the standard coordinates X, Y than to the plate coordinates, measured on an engine, of photographic images. The standard coordinates will be compared eventually with the plate coordinates, and therefore it is logically immaterial whether one corrects the plate coordinates, or allows in the standard coordinates, for distortion and refraction; and the latter procedure is easier. Atmospheric refraction moves the observed position of a star towards the zenith by an angle

$$z = u \tan z \tag{8}$$

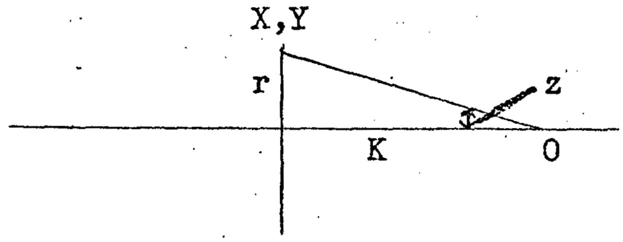
where z is the zenith distance and *

$$u = \frac{983b'}{460+t} \tag{9}$$

where b' is the barometer reading at the camera in inches, and t is the temperature of the air near the camera in degrees Fahrenheit. These formulae represent observed refractions correctly within a second of arc for zenith distances less than 75°, and for distances under 20° are considerably more accurate. At Aberdeen, u is about 58" and this value can be taken as standard, with an accuracy sufficient for present purposes (for reasons which will be obvious later). It is easily verified that within 20° of the zenith, it is immaterial whether the refracted or unrefracted tan z is used in formulae (8), for the difference of refraction thereby introduced is smaller than 0".01.

Denote by r the distance $(X^2 + Y^2)^{1/2}$ of the point X, Y from the origin on the standard coordinate plane.

* Comstock, Sidereal Messenger, April, 1890.



The effect of refraction is to diminish $\underline{z}$ by ur/K, and thus to diminish

$$r = K \tan \underline{z}$$

by

$$dr = -K(\sec^2 \underline{z})\ ur/K$$
$$= -ur(1 + r^2/K^2).$$

The first term, linear in $\underline{r}$, amounts merely to a change of scale and can be ignored, since the reduction procedure will automatically allow for scale factors. We are left with the third-order term

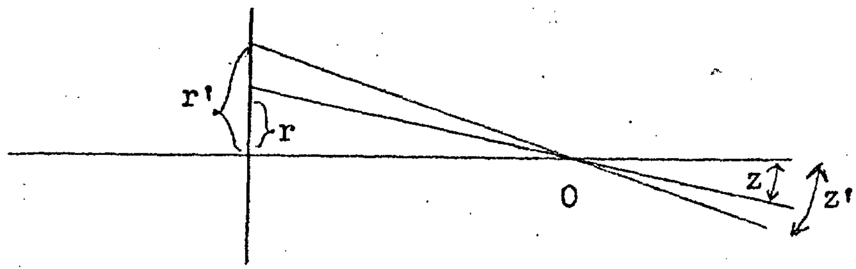
$$dr = -ur^3/K^2 \qquad (10)$$

where u must be expressed in radians. With u = 58", K = 10, (10) is

$$dr = -0.000002812\ r^3 \qquad (11)$$

This amounts only to 2".7 at the edge of a Goerz plate, and changes only 0".1 for a change of one inch in the barometer, or of 17° F. It is for the preceding reason that a constant value of 58" may be adopted for u at the Proving Ground, with sufficient accuracy.

The effect of distortion is similar, and can involve only odd powers of $\underline{r}$. The diagram below shows the nodal point of the lens at O; and a ray with an angle of incidence of $\underline{z}$, before entering the lens, and with an angle of incidence $\underline{z'}$ after traversing the lens -- the difference $(\underline{z'} - \underline{z})$ being the angular distortion. It is clear that $\underline{r'}$ is an odd function of $\underline{z}$, and hence that the linear distortion $(\underline{r'} - \underline{r})$ must be an odd function of $\underline{r}$ involving only odd powers of $\underline{r}$.



For the same reason as in the case of refraction, the terms linear in r can be ignored, being absorbed in the scale factor, and we are left only with cubic and higher odd powers of r, thus:

$$dr = A'r^3 + B'r^5 + \ldots$$

In combination with the cubic terms in refraction, the combined effect is therefore

$$dr = A''r^3 + B'r^5 + \ldots \tag{12}$$

The constants A" and B' (B' will probably be trivial) must be found once and for all empirically, by the method to be described shortly, for each camera lens. Although A" includes refraction, dependent on atmospheric conditions, changes in the part of A" that arises from refraction are, as we have seen, minute and ignorable and thus A" and B' may be regarded as constant for practical purposes. The correction of the X and Y of a star, to allow for dr given by (12), is carried out as follows. Set $X = r \cos \theta$, $Y = r \sin \theta$, where θ is an azimuthal angle (on the standard coordinate plane) that is not altered by refraction and distortion. Then

$$dX = \cos \theta \, dr; \quad dY = \sin \theta \, dr$$

and since

$$r^2 = X^2 + Y^2$$

one has

$$\begin{aligned}(dX)_d &= X A'' (X^2 + Y^2) \left[1 + \frac{B'}{A''} (X^2 + Y^2)\right] \\ (dY)_d &= Y A'' (X^2 + Y^2) \left[1 + \frac{B'}{A''} (X^2 + Y^2).\right]\end{aligned} \tag{XIII}$$

These equations enable the computer readily to correct the values of X and Y already computed for each star, thus:

$$X' = X + (dX)_d \qquad Y' = Y + (dY)_d \tag{XIV}$$

where the X' and the Y' are standard coordinates, with refraction and distortion allowed for.

6. Determination of the Constants A" and B' (Distortion and Stellar Refraction. To determine A" and B' for a camera, it is to be very carefully levelled and then a star-photograph is to be taken. Compute the uncorrected standard coordinates, X and Y, for each star. A "star" can, of course, be a break in a star-trail, or the average of a number of breaks in the same trail. By the method later to be described, find by least squares the plate constants a, b, c, d, e, f in the equations

$$a + b\,x + c\,y = X$$

$$d + e\,x + f\,y = Y$$

where x and y are plate coordinates. Then find the constants A" and B' from the residuals by least squares, using the observational equations

$$C'X + A''X\,(X^2 + Y^2) + B'X\,(X^2 + Y^2)^2 = a + bx + cy - X$$

$$C'Y + A''Y\,(X^2 + Y^2) + B'Y\,(X^2 + Y^2)^2 = d + ex + fy - Y$$

The unknowns are C', A", and B'; where C', although it must be included in the least squares solution, is of no interest when found. The stars should cover the plate as fully and as uniformly as possible, for best results, and the present determination need be carried out only once and for all, for each camera.

7. Determination of the Plate Constants. The equations

$$a + bx + cy = X' \qquad d + ex + fy = Y' \tag{XV}$$

relate the measured plate coordinates, x and y, to the corrected standard coordinates X' and Y'. The equations (XV) allow for scale factor, for rotation of the plate in the measuring engine, for lack of perpendicularity of the ways of the measuring engine, for a difference between the scales of the x and y screws, and for arbitrary zero-points of x and of y. Three stars will determine the plate constants a, b, c, d, e, and f; more than three stars will overdetermine the plate constants and a least squares solution should be made. The solution with n stars, 1, 2, 3, ... n, proceeds as follows. Let $\overline{x}$ be the mean of the x's, $\overline{y}$ be the mean of the y's, and

$$\begin{aligned} x' &= x - \overline{x} \\ y' &= y - \overline{y}. \end{aligned} \tag{XVI}$$

Then the observational equations are

$$(a + b\overline{x} + c\overline{y}) + bx'_i + cy'_i = X'_i$$

$$(d + e\overline{x} + f\overline{y}) + ex'_i + fy'_i = Y'_i$$

and the normal equations (the unknowns being $(a + b\overline{x} + c\overline{y} = a')$, b, c, $(d + e\overline{x} + f\overline{y} = d')$, e, and f) are

$$\begin{array}{cccc} a' & b & c & = \\ n & 0 & 0 & X' \\ & x'^2 & x'y' & x'X' \\ & & y'^2 & y'X' \end{array}$$

with similar equations in Y' for d', e, and f. To solve these, compute

$$\Delta = \left[x'^2\right]\left[y'^2\right] - \left[x'y'\right]^2$$

$$b_i = \left\{x'_i \left[y'^2\right] - y'_i \left[x'y'\right]\right\}/\Delta \tag{XVII}$$

$$c_i = \left\{y'_i \left[x'^2\right] - x'_i \left[x'y'\right]\right\}/\Delta$$

and check these quantities by the equations

$$
\begin{aligned}
[b_i] &= 0 & [c_i] &= 0 \\
[b_i x_i] &= 1 & [b_i y_i] &= 0 \\
[c_i x_i] &= 0 & [c_i y_i] &= 1.
\end{aligned}
\qquad \text{(XVIII)}
$$

Then

$$
\begin{aligned}
a &= (1/n)\,[X'_i] - b\bar{x} - c\bar{y} & d &= (1/n)\,[Y'_i] - e\bar{x} - f\bar{y} \\
b &= [b_i X'_i] & e &= [b_i Y'_i] \\
c &= [c_i X'_i] & f &= [c_i Y'_i]
\end{aligned}
\qquad \text{(XIX)}
$$

and when only three stars are used, equations (XV) may be used for final checks and will be satisfied exactly. With more than three stars, equations (XIX) must be repeated to provide a check; another check is that different plates will have nearly the same plate constants unless the cameras have been disturbed between exposures.

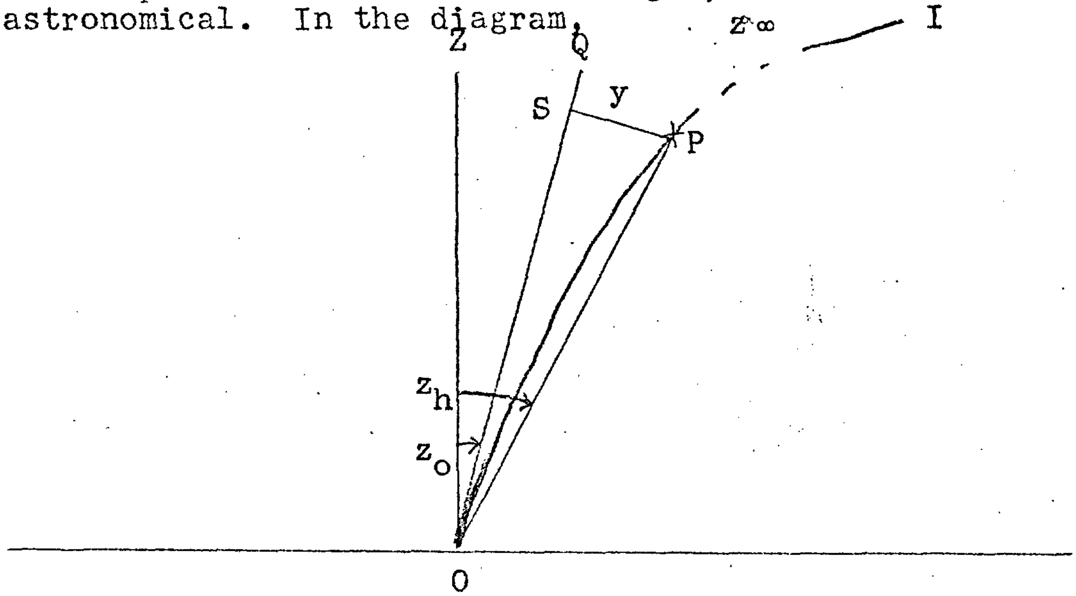
When there are only three stars, the preceding solution holds, but it is quicker merely to solve the six simultaneous equations (XV) for a, b, c, d, e, and f by successive eliminations. The equations group into two sets of three.

8. Determining the Standard Coordinates of the Aerial Point. One has merely to insert in equation (XV) the plate coordinates of any aerial point in order to find its standard coordinates, X' and Y'. These are to be corrected by applying dX and DY given by (XIII), thus:

$$
\begin{aligned}
X &= X' - (dX)_d \\
Y &= Y' - (dY)_d
\end{aligned}
\qquad \text{(XX)}
$$

where (XIII) may be entered, to sufficient accuracy, with X' and Y' instead of X, Y. These standard coordinates have next to be corrected for the difference between astronomical refraction, and the smaller refraction that affects the ray to the airplane immersed in the atmosphere.

9. <u>The Refraction Affecting the Aerial Point</u>. If the aerial point were at infinite height, its refraction would be astronomical. In the diagram,



O is the camera; Z is the zenith; OPI is the ray from the aerial point; P is the aerial point; OQ is the tangent to the ray at O; z_o is the zenith distance of the ray at the camera; z_h is the zenith distance of the straight line OP; z_∞ is the zenith distance of the ray OPI extended to infinity; h is altitude above the camera. The normal distance from OQ to the ray OPI at the distance s from O is y. Now if z is the variable zenith distance along the ray and if μ is the index of refraction of air, one has

$$\frac{dz}{ds} = - \sin z \frac{d\mu}{\mu dh} \tag{21}$$

or since

$$\frac{dh}{ds} = \cos z,$$

one has

$$\cot z \, dz = - \frac{d\mu}{\mu}$$

whence

$$\sin z = (\mu_o/\mu) \sin z_o \tag{22}$$

where μ_o is the index of refraction of the air near the camera.

Since $z - z_o = \Delta z$ is small, one has

$$\Delta z = \frac{\mu_o - \mu}{\mu} \tan z_o$$

whence

$$\begin{aligned} y &= \int_o^s z\, ds \\ &= \tan z_o \sec z_o \int_o^h \frac{\mu_o - \mu}{\mu}\, dh \end{aligned} \tag{23}$$

and

$$\begin{aligned} z_h - z_o &= y/h \sec z_o \\ &= \tan z_o \frac{1}{h} \int_o^h \frac{\mu_o - \mu}{\mu}\, dh. \end{aligned} \tag{24}$$

We have

$$\mu = 1 + (\mu_o - 1) \rho/\rho_o$$

where ρ is the air density, so that

$$z_h - z_o = (\mu_o - 1) \tan z_o \frac{1}{h} \int_o^h \frac{1 - (\rho/\rho_o)}{1+(\mu_o-1)\rho/\rho_o}\, dh. \tag{25}$$

The denominator is closely enough equal to unity, since $\mu_o - 1$ is small; and thus, very nearly,

$$z_h - z_o = (\mu_o - 1) \tan z_o \left[1 - \frac{1}{h} \int_o^h (\rho/\rho_o)\, dh\right]. \tag{26}$$

The astronomical refraction is $z_\infty - z_o$, given by setting $h = \infty$ in (26):

$$z_\infty - z_o = (\mu_o - 1) \tan z_o ,$$

which agrees with (8) if $u = \mu_o - 1$ is taken as 58" of arc, or as 0.0002812 radians. Thus the difference between the astronomical refraction and the actual is

$$z_\infty - z_h = (1/1000) \tan z_o \, f(h) \tag{27}$$

where

$$f(h) = 0.2812 \frac{1}{h} \int_o^h (\rho/\rho_o)\, dh.$$

The value of f(h) up to 130,000 feet has been computed from the annual means of the observed values of ρ/ρ_o, obtained by sounding balloons, and published by Humphreys:*

h (ft)	f(h)	h (ft)	f(h)	h (ft)	f(h)
0	.281	45000	.144	90000	.083
5000	.258	50000	.135	95000	.079
10000	.239	55000	.126	100000	.075
15000	.222	60000	.118	105000	.072
20000	.206	65000	.110	110000	.069
25000	.191	70000	.104	115000	.066
30000	.178	75000	.098	120000	.063
35000	.166	80000	.093	125000	.061
40000	.155	85000	.087	130000	.059

* W. J. Humphreys, Physics of the Air, Franklin Institute, 1920.

The above values apply to a barometer reading of 30 inches, and a temperature of 48° F, at the camera. In accordance with equation (9), they should be lowered by 1% for each 5° F by which the temperature exceeds 48° F, and increased by 1% for each 0.3 inches by which the pressure exceeds 30 inches. The altimeter will furnish values of <u>h</u> accurate enough for finding f(h) by means of the preceding table and corrections.

By the argument of section 5, we allow for the difference of refractions by applying the correction dr, where

$$1000\ dr = -K(1 + r^2/K^2)\ r\ f(h)/K$$

or by applying the corrections $(dX)_r$, $(dY)_r$, where

$$\begin{aligned} 1000\ (dX)_r &= f(h)\ X \left[1 + (X^2 + Y^2)/K^2\right] \\ 1000\ (dY)_r &= f(h)\ Y \left[1 + (X^2 + Y^2)/K^2\right] \end{aligned} \qquad \text{(XXVIII)}$$

to the X and Y of the aerial point, in order to find X", Y":

$$\begin{aligned} X'' &= X - (dX)_r \\ Y'' &= Y - (dY)_r. \end{aligned} \qquad \text{(XXIX)}$$

10. <u>The Direction Ratios to Aerial Points</u>. With respect to the set of axes X, Y, Z, defined in section 3 (X to the east, Y to the north, and Z to the zenith), having an origin at either camera: The geometrical coordinates of the intersection, of the standard coordinate plane with the straight line from the camera to the aerial point, are simply X", Y", K. Hence the direction ratios of the line from the camera to the aerial point are merely

$$X'' : Y'' : K. \qquad \text{(XXX)}$$

It will be henceforth necessary to distinguish between the two cameras, which will be denoted as camera 1 and camera 2.

The ratios for camera 1 will be denoted by $X''_1 : Y''_1 : K$ and those for camera 2 by $X''_2 : Y''_2 : K$. It will be necessary to transform these direction ratios to a standard terrestrial system of reference. Because of the curvature of the earth, the X, Y, Z systems of the two cameras will not be parallel to each other. If the standard system of reference were chosen to coincide with the X, Y, Z system of one of the cameras, some computation would be saved; but then the direction of gravity over the bomb-trajectory would not coincide with the standard z-direction. We discuss the transformations as though the standard reference system differed from the X, Y, Z systems of both cameras; if it is identical with either of the latter then the following results and procedure will still hold.

Let P_o denote the origin, at latitude φ_o, of a standard terrestrial reference system, ξ, η, ζ. The latitudes of cameras 1 and 2 are denoted by φ_1 and φ_2, and their longitudes measured to the west from P_o are denoted by α_1 and α_2. X_o, Y_o, Z_o are a set of axes through P_o as origin, such that Z_o is vertical, Y_o is to the north, and X_o is to the east. X_1, Y_1, Z_1 and X_2, Y_2, Z_2 are similar sets of axes through cameras 1 and 2 as origins; these three sets of axes are of course not parallel to each other. The standard reference system differs from the X_o, Y_o, Z_o system through a rotation about the Z_o-axis, which coincides with the ζ-axis. The η-axis has an azimuth at P_o of λ, measured from north to east; the ξ-axis has an azimuth at P_o of $(\lambda + 90°)$. We also make use in this discussion of the hour-angle declination axes x'_o, y'_o, z'_o; with P_o as origin, with x'_o lying in the equator and meridan, y'_o pointing to the east, and z'_o pointing to the north celestial pole. x'_1, y'_1, z'_1 and x'_2, y'_2, z'_2 are similar axes with origins at cameras 1 and 2, respectively.

It is necessary to find the cosines of the angles between the ξ, η, ζ axes and the X_1, Y_1, Z_1 axes; and between the ξ, η, ζ axes and the X_2, Y_2, Z_2 axes. The cosines of the angles between the x_o and the ξ systems are

	X_o	Y_o	Z_o
ξ	$a_{11} = \cos\lambda$	$a_{12} = -\sin\lambda$	$a_{13} = 0$
η	$a_{21} = \sin\lambda$	$a_{22} = \cos\lambda$	$a_{23} = 0$
ζ	$a_{31} = 0$	$a_{32} = 0$	$a_{33} = 1$

The cosines relating the X_o system to the x'_o system are

	x'_o	y'_o	z'_o
X_o	$b_{11} = 0$	$b_{12} = 1$	$b_{13} = 0$
Y_o	$b_{21} = -\sin\varphi_o$	$b_{22} = 0$	$b_{23} = \cos\varphi_o$
Z_o	$b_{31} = \cos\varphi_o$	$b_{32} = 0$	$b_{33} = \sin\varphi_o$

,

the cosines relating the x'_1 system to the x'_o system are

	x'_1	y'_1	z'_1
x'_o	$c_{11} = \cos\alpha_1$	$c_{12} = \sin\alpha_1$	$c_{13} = 0$
y'_o	$c_{21} = -\sin\alpha_1$	$c_{22} = \cos\alpha_1$	$c_{23} = 0$
z'_o	$c_{31} = 0$	$c_{32} = 0$	$c_{33} = 1$

,

and the scheme relating the systems x'_1 and X_1 is

	X_1	Y_1	Z_1
x'_1	$d_{11} = 0$	$d_{12} = -\sin\varphi_1$	$d_{13} = \cos\varphi_1$
y'_1	$d_{21} = 1$	$d_{22} = 0$	$d_{23} = 0$
z'_1	$d_{31} = 0$	$d_{32} = \cos\varphi_1$	$d_{33} = \sin\varphi_1$

.

From these we compute in succession the arrays

	x'_o	y'_o	z'_o
ξ	e_{11}	e_{12}	e_{13}
η	e_{21}	e_{22}	e_{23}
ζ	e_{21}	e_{22}	e_{23}

where

$$e_{ij} = \sum_{k=1}^{3} a_{ik}\, b_{kj} \quad ,$$

	x'_1	y'_1	z'_1
ξ	f_{11}	f_{12}	f_{13}
η	f_{21}	f_{22}	f_{23}
ζ	f_{31}	f_{32}	f_{33}

where

$$f_{ij} = \sum_{k=1}^{3} e_{ik}\, c_{kj} \quad ,$$

and

	X_1	Y_1	Z_1
ξ	g_{11}	g_{12}	g_{13}
η	g_{21}	g_{22}	g_{23}
ζ	g_{31}	g_{32}	g_{33}

where

$$g_{ij} = \sum_{k=1}^{3} f_{ik}\, d_{kj} \,.$$

The preceding computations are easily performed if actual numbers are inserted in the cells, and lead to the numerical array, g_{ij}, relating the X_1, Y_1, Z_1 system to the standard system. Then the numbers and headings should be

	X''_1	Y''_1	+
ξ	g_{11}	g_{12}	Kg_{13}
η	g_{21}	g_{22}	Kg_{23}
ζ	g_{31}	g_{32}	Kg_{33}

written down on a heavy card, headed "Camera 1".

In precisely the same way, replacing camera 1 by camera 2, and thus φ_1 by φ_2 and α_1 by α_2, one obtains the scheme, which should be written down numerically on a heavy card, headed "Camera 2".

	X''_2	Y''_2	+
ξ	h_{11}	h_{12}	Kh_{13}
η	h_{21}	h_{22}	Kh_{23}
ζ	h_{31}	h_{32}	Kh_{33}

The transformation of the direction ratios $X''_1 : Y''_1$ K to the system of standard terrestrial axes ξ, η, ζ is merely

$$A_1 = X''_1 \, g_{11} + Y''_1 \, g_{12} + (Kg_{13})$$

$$B_1 = X''_1 \, g_{21} + Y''_1 \, g_{22} + (Kg_{23}) \qquad \text{(XXXI)}$$

$$C_1 = X''_1 \, g_{31} + Y''_1 \, g_{32} + (Kg_{33})$$

where $A_1 : B_1 : C_1$ are the direction ratios, of the straight line from camera 1 to the aerial point, expressed in the ξ, η, ζ system. Likewise, the direction ratios $X''_2 : Y''_2 : K$ from camera 2 are transformed to the ratios $A_2 : B_2 : C_2$ in the ξ, η, ζ system by the equations

$$A_2 = X''_2 \, h_{11} + Y''_2 \, h_{12} + (Kh_{13})$$

$$B_2 = X''_2 \, h_{21} + Y''_2 \, h_{22} + (Kh_{23}) \qquad \text{(XXXII)}$$

$$C_2 = X''_2 \, h_{31} + Y''_2 \, h_{32} + (Kh_{33}).$$

Using the coefficients written on the cards, a good computer should be able to transform a single set of direction ratios $X'' : Y'' : K$ in little more than one minute; the coefficients, which depend only on the camera positions and on the standard reference system that is used, are mere constants.

If the origin P_o of the reference system ξ, η, ζ coincides with either camera, and if the η-axis runs exactly north from that camera, then no transformation is necessary of the ratios $X'' : Y'' : K$ obtained from that camera, and one has for that camera $A = X'' : B = Y'' : C = K$. Thus some saving of computation-time can result from such a choice of a standard reference system.

11. <u>The Coordinates of the Aerial Point</u>. Denote the ξ, η, ζ coordinates of camera 1 by a_1, b_1, c_1; and those

of camera 2 by a_2, b_2, c_2. Denote the direction ratios of the rays from the two cameras to the aerial point, in the x, y, z system, by $A_1 : B_1 : C_1$ and $A_2 : B_2 : C_2$ as in section 10. Then the coordinates x, y, z of the aerial point are given by the equations

$$
\begin{aligned}
\xi &= a_1 + A_1\, p = a_2 + A_2\, m \\
\eta &= b_1 + B_1\, p = b_2 + B_2\, m \\
\zeta &= c_1 + C_1\, p = c_2 + C_2\, m
\end{aligned}
\qquad \text{(XXXIII)}
$$

where p and m are found by solving the observational equations, by least squares,

$$
\begin{array}{ccc}
p & m & = \\
A_1 - A_2 & & a_2 - a_1 \\
B_1 - B_2 & & b_2 - b_1 \\
C_1 - C_2 & & c_2 - c_1 .
\end{array}
\qquad (34)
$$

Compute

$$
\begin{aligned}
F &= A^2_1 + B^2_1 + C^2_1; \\
G &= A^2_2 + B^2_2 + C^2_2; \\
H &= A_1A_2 + B_1B_2 + C_1C_2; \\
I &= (a_2-a_1)\, A_1 + (b_2-b_1)\, B_1 + (c_2-c_1)\, C_1; \\
J &= (a_2-a_1)\, A_2 + (b_2-b_1)\, B_2 + (c_2-c_1)\, C_2; \\
K &= FG - H^2.
\end{aligned}
\qquad \text{(XXXV)}
$$

Then

$$p = (GI - HJ)/K \qquad \text{(XXXVI)}$$

$$m = (HI - FJ)/K$$

and these computations should be checked by computing residuals of the three observational equations, and seeing that the sum of their products with the coefficients of p should vanish, and that the sum of their products with the coefficients of m should also vanish. This checks the least squares solution. Then from (XXXIII), one finds two values of ξ, of η, and of ζ; the agreement indicates the accuracy of the whole work, and for final values the averages of the two estimates of each quantity, ξ, η, and ζ, should be taken.

This completes the description of the method of finding the position of an aerial point, from photographs taken from two vertical cameras, by the use of star-images. We can call the preceding method the "long method".

12. Summary of the "Long Method"; Directions for Computers; Estimated Computation-Times. The preceding exposition has been long, but the method is fairly rapid. It may be summarized as follows. At the outset, a study should be made as described in section 6 to find the distortion and third-order refraction terms for the two cameras. At the outset, also, when the piers are set out, one should prepare the two cards, each with nine constants, described in section 10. Then the procedure with two plates, one from each camera, in a particular range-bombing program is as follows: Measure the plate coordinates, (x, y), of each comparison star and aerial point on both plates. While this is being done, compute the direction cosines L_o, M_o, N_o of the comparison stars by equations (I) and (III) -- unless they are already available from the computations of a previous range bombing. Correct these for nutation, etc., by equations (IV) and compute their standard coordinates for each

camera by equations (VI) and (VII). Apply the checks described in the text, at all stages. Apply the corrections given by equations (XIII) to all standard coordinates, by equations (XIV). Determine the plate constants from the comparison stars, using equations (XVI) through (XIX). Determine the standard coordinates of the aerial points by equations (XV), correct them by equations (XX) and (XXIX), and compute their direction ratios by equations (XXXI) and (XXXII), using the two cards. Finally compute the space coordinates of the aerial points by using equations (XXXV), (XXXVI), and (XXXIII). The writer estimates, very roughly, a computation-time of from four to eight hours for a single computer when there are three comparison stars, and three aerial points to locate. With a suitable arrangement of the work, three computers should accomplish it in about two to four hours. These times are for experienced computers; mediocre ones would probably take longer.

13. Rapid Method Without Using Stars. If the cameras are very rigidly mounted on well-settled piers (protected further from diurnal or other rapid temperature changes, and not moving with the tides), it may prove possible to avoid the employment of star-images on some of the plates. For this "short method" to be possible, fiducial marks must be impressed on the plates by marking devices that are absolutely fixed with respect to the lenses and piers. Further, the lens of each camera must be absolutely free from rattle in its cell, and the camera itself must be of the firmest and most rigid construction. These requirements are unnecessary when star images are used, as in the "long method".

Using stars as already described, it is necessary to find the plate constants a, b, c, d, e, and f (as explained in paragraph 7) for some plate which we shall call the "standard" plate, which need not photograph the aerial point. Coordinates of points on the standard plate will be written thus: x, y. Then the plate constants relate x and y to directions in space. Suppose now that we measure the coordinates x, y for a number of fiducial marks impressed on the standard plate by the camera, and then take another plate, (the "observing" plate), on which the same marks are impressed, without disturbing the camera in any way between exposures. We can denote coordinates on the observing plate by (x', y') and* we can measure the

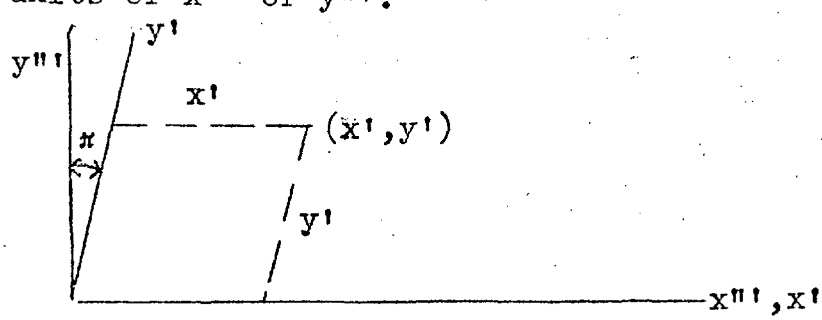
* The plate coordinates (x', y') are not the space coordinates (x', y', z') of Section 3.

coordinates (x', y') of the fiducial marks and of the aerial points on the observing plate. No stars need appear on the observing plate. Then it is obvious that if there are three or more fiducial marks on both plates, we can find from the coordinates x, y and x', y' of the fiducial marks, the six coefficients in two linear equations, like the equations (XV), which relate the coordinates x', y' to the coordinates x, y. Such relations will automatically allow for changes of scale in the measuring engine, for lack of perpendicularity of the ways, for rotations of the plates in the measuring engine, and for differences between the scales of the two screws of the engine. They will also automatically allow for thermal expansions of the plates, and of the camera. Thus from the x', y' of an aerial point on the observing plate, we can compute its x and y by the linear relations so found, and thence deduce the directions $X'' : Y'' : K$ of the line to the aerial point, and thence finally obtain the space coordinates ξ, η, ζ of the aerial point, just as though the aerial point had been photographed on the standard plate itself. Care must be taken to allow for differences between the refractions influencing the two plates, if they are taken on different days; but the procedure is straightforward and it is left to the reader to develop the formulae applicable to the case of three or more fiducial marks. Unfortunately, with the Goerz cameras of the Aberdeen Proving Ground, there are only two fiducial marks on each camera. We shall discuss this case with care.

Let us inquire as to what differences there can be between the standard plate and the observing plate, apart from changes of refraction, when the camera has not been disturbed. The <u>x'</u>'s and <u>y'</u>'s can differ from the <u>x</u>'s and <u>y</u>'s through changes in temperature, either at the times of measurement or at the times of exposure, or both. Changes of temperature, whether affecting the plates through their glass, or through swelling or contracting of the camera, or through changes in the pitches of the measuring screws, can at most amount to a simple change in scale. Thus we should expect the two plates to differ in scale. If the plates are measured on different engines, then the pitches of the two x-screws can differ in ratio from the pitches of the two y-screws. This must be avoided, when only two fiducial marks are available, by using the same engine for measuring both plates. Further, we must take care in the present "short" method not to use an engine which has only one screw, and which depends for measurements of x and y upon a rotation of the plate through 90°

between measuring the x's and measuring the y's. It is essential to use an engine with two screws, very nearly at right angles. With a single-screw engine the angle between the axes of measurement will not only never be 90°, but it will vary from plate to plate according to the accuracy with which the x and y dispositions can be reproduced. While that has no evil effect when there are three fiducial marks, it introduces insuperable difficulties when there are but two marks. With a two-screw engine, the angle between the ways may not be 90° exactly, and in general it will not be 90°; but whatever the angle is, it will remain constant from plate to plate unless the engine is subjected to abuse. Moreover, the ratios between the scales of the two screws will remain constant from plate to plate, since the screws are always made of the same metal. Another source of difference between (x', y') and (x, y) arises from the inevitable small differences between the orientations of the plates in the measuring engine when the plates are measured. These cannot be entirely avoided. Finally, the x''s and y''s can differ from the x's and y's through the two systems' having different zero-points. We are left, therefore, when all precautions have been taken, with the effects of differences of scale, of different orientations in the measuring engine, and of different zero-points. We discuss the relation between x', y' and x, y in the following section, and show how the effects of the preceding differences can be automatically eliminated.

14. Short Method - The Relation of the Standard Plate to the Observing Plate. In the diagram below, we have drawn the x' and y' axes, and also a set of orthogonal axes (x"', y"') such that x"' is parallel to x'. We make the origins coincide here and in the following transformations, since zero-point constants are easily inserted later on. One unit of x' will be b_s units of x"' or y"', and the angle π measures the lack of perpendicularity of the ways of the engine, x' and y'. A unit of the y' screw is c_s units of x"' or y"'.



Then

$$x''' = b_s x' + c_s y' \sin\kappa$$

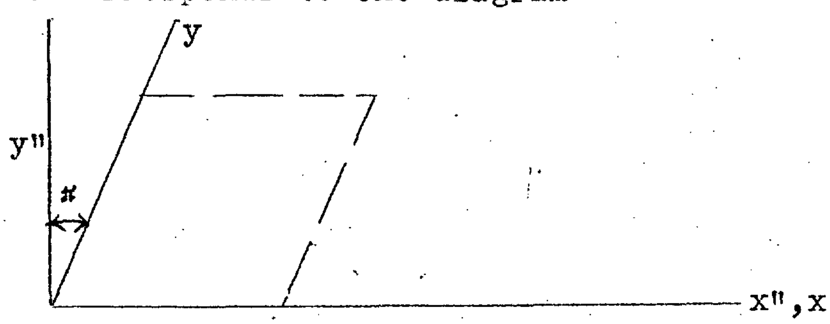
$$y''' = c_s y' \cos\kappa$$

The total change of scale and the effect of the different orientations of the plates in the engine, can be represented by a transformation from (x''', y''') to (x'', y'') where the latter coordinates differ from the former through a rotation ω and through a scale factor $\underline{s}$. Thus

$$x'' = sx''' \cos\omega - sy''' \sin\omega$$

$$y'' = sx''' \sin\omega + sy''' \cos\omega.$$

Finally, the measurement of the standard plate in the engine corresponds to the diagram



where the relation of (x'', y'') to (x, y) is the same as the relation of (x''', y''') to (x', y'). Thus

$$x = \frac{1}{b_s}(x'' - y'' \tan\kappa)$$

$$y = \frac{y''}{c_s \cos\kappa}$$

It will be noticed that we have assumed the same relative pitches of the x and y screws for the two plates, and the same error of the ways $\varkappa$; and that changes of scale from one plate to the other have been absorbed in the constant s. Combining the preceding linear equations, and adding zero-point constants, one finds that in general

$$x = \Gamma + s(\cos\omega - \sin\omega \tan\varkappa)\, x' - \frac{c_s s}{b_s} \sin\omega \sec\varkappa \,.\, y'$$

$$y = \Pi + \frac{b_s s}{c_s} \sin\omega \sec\varkappa \,.\, x' + s(\cos\omega + \sin\omega \tan\varkappa)\, y'$$

which can be written

$$x = \Gamma + \Lambda x' + \Omega y' - \left[+\Lambda \tan\omega \tan\varkappa \,.\, x' + \Omega \left(1 - \frac{c_s}{b_s}\right) y' \right]$$

$$y = \Pi - \Omega x' + \Lambda y' + \left[+ \Omega \left(1 - \frac{b_s}{c_s}\right) x' + \Lambda \tan\omega \tan\varkappa \,.\, y' \right]$$

where Λ has been written for $s \cos\omega$ and Ω has been written for $-s \sin\omega \sec\varkappa$.

Since $\varkappa$ is a very small angle, and ω is small, while the scale factors s, b_s, and c_s are all very close to unity, it follows that Λ is nearly unity while Ω is approximately equal to $-\sin\omega$. The ratios of scales b_s/c_s is very close to unity for any well-made engine; certainly not differing from unity by more than one part in 10,000. The angle $\varkappa$ for a well-made engine should be less than one minute, so $\tan\varkappa$ is less than 0.0003. If (by careful alinement of the observing plate in the measuring engine so that x, y and x', y' are nearly the same for both fiducial points) we can keep ω less than 1°, then the coefficients of x' and y' in the square bracketed second-order terms are all smaller than 0.000005 and are ignorable. We are then left with

$$\begin{aligned} \Gamma + \Lambda x' + \Omega y' &= x \\ \Pi - \Omega x' + \Lambda y' &= y \end{aligned} \qquad \text{(XXXVII)}$$

with an accuracy sufficient to yield directions that will be correct to better than a second of arc. Here Γ, Λ, Ω, Π are unknown constants to be determined from the measured coordinates (x, y) and (x', y') of the fiducial marks. It is easy to aline both plates, in the measuring engine, similarly to within one degree of angle. We may call Γ, Λ, Ω, Π the "observing constants".

15. <u>Short Method - Finding the Observing Constants.</u> It is readily found that

$$\begin{aligned} \Lambda &= \left[(x'_1 - x'_2)(x_1 - x_2) + (y'_1 - y'_2)(y_1 - y_2) \right] / E \\ \Omega &= \left[(y'_1 - y'_2)(x_1 - x_2) - (y_1 - y_2)(x'_1 - x'_2) \right] / E \\ \Gamma &= x_1 - \Lambda x'_1 - \Omega y'_1 \\ \Pi &= y_1 + \Omega x'_1 - \Lambda y'_1 \end{aligned} \qquad \text{(XXXVIII)}$$

where

$$E = (x'_1 - x'_2)^2 + (y'_1 - y'_2)^2.$$

The observing constants, Γ, Λ, Ω, Π may be checked by the equations (XXXVII). Here (x'_1, y'_1) and x'_2, y'_2) are the measured coordinates of the fiducial marks 1 and 2, respectively, on the observing plate; and (x_1, y_1) and (x_2, y_2) are the measured coordinates of the fiducial marks on the standard plate. As has been pointed out, both plates must be measured in the same two-screw measuring engine, after having been inserted in nearly parallel orientations. The observing constants are, of course, relative to the particular observing plate, and particular standard plate. It is entirely proper to use, as a standard plate, the average of two or more star-plates taken at nearly the same time; then one has to use (later) the average of the air temperatures and pressures when correcting for refraction on the standard plate.

16. Short Method - The Direction Ratios to Aerial Points. Let the plate coordinates of an aerial point be (x', y') as measured on the observing plate. Insert these values in equations (XXXVII) and thereby obtain (x, y). Now insert (x, y) in the left-hand members of equations (XV), involving the plate constants of the standard plate, and thereby find the standard coordinates (X', Y'), of the aerial point. The original values (x', y') contained the effects of refraction of the aerial point, and lens distortion. The refraction in question is that appropriate to the date of the observing plate. It is characteristic of the transformations (XV) that they automatically take out the first order astronomical refraction for the date of the standard plate. Therefore, the standard coordinates (X', Y') contain the refraction of the aerial point, distortion, and minus the first order astronomical refraction of the standard plate. If we subtract from X' and Y' the small quantities $(dX)_d$ and $(dY)_d$ given by (XIII) entering these equations with (X', Y'), we are left with standard coordinates free from distortion, and affected by minus the full astronomical refraction of the standard plate, and plus the refraction of the aerial point on the date of the observing plate. If we now apply the difference (astronomical refraction on date of standard plate) minus (astronomical refraction on date of observing plate) and apply further the difference (astronomical refraction) minus (refraction to height h) appropriate to the date of the observing plate, we shall be left with standard coordinates (X'', Y'') fully corrected for distortion and refraction and comparable in all respects with the standard coordinates (X'', Y'') furnished by equations (XXIX) of the long method. The difference between the astronomical refractions on the two dates is obtained readily from equations (8) and (9), and thus we have for any aerial point

$$X'' = X' - (dX)_d + (u_{ob} - u_{st})\, X' - (dX)_r$$

$$Y'' = Y' - (dY)_d + (u_{ob} - u_{st})\, Y' - (dY)_r \qquad \text{(XXXIX)}$$

where

$$u = \frac{0.004766\ b'}{460 + t}$$

and where u_{ob} refers to the observing plate, u_{st} to the standard plate. Here <u>b'</u> is the air pressure at the camera in inches of mercury, and <u>t</u> is the air temperature in degrees Fahrenheit.

Thus the short procedure goes as follows: Insert the plate coordinates (x', y') of the aerial point, as measured on the observing plate, into equations (XXXVII) and thus obtain (x, y). Insert (x, y) into the left-hand members of equations (XV) appropriate to the observing plate, and thus find (X', Y'). Correct (X', Y') by equations (XXXIX) to find (X'', Y''). In equations (XXXIX), find $(dX)_d$ and $(dY)_d$ from equations (XIII), ignoring the difference between (X, Y) and (X', Y'), and similarly find $(dX)_r$ and $(dY)_r$ from equations (XXXVIII) and the associated table.

17. <u>Short Method - The Coordinates of the Aerial Point</u>. The quantities (X'', Y'') just obtained are precisely comparable to the same quantities in the long method, and from there on the reduction goes just as in the long method, described in sections 10 and 11.

18. <u>Summary of the "Short" Method; Directions for Computers; Estimated Computation-Times</u>. Standard plates containing star-images must have been previously obtained from both cameras; then the cameras, without being in any way disturbed, photograph the aerial points on observing plates. The comparison star-images and fiducial marks must be measured on the standard plates; and the plate constants must be obtained as in the "long" method for the standard plates. The fiducial marks and the images of the aerial points are measured on the observing plates,

by the use of the same engine that was used to measure the standard plates. The standard plates and observing plates must be inserted in the measuring engine (which must be of the two-screw type) approximately parallel to each other. Find the observing constants by equations (XXXVIII), and then by equations (XXXVII), compute standard plate coordinates (x, y) for each aerial point. Compute standard coordinates (X', Y') by equations (XV). Correct these by equations (XXXIX), obtaining the corrected standard coordinates (X", Y") of the aerial points. By formulae (XXXI) and (XXXII), compute the direction ratios, using the two cards. Finally compute the space coordinates ξ, η, ζ of the aerial points by using equations (XXXV), (XXXVI), and (XXXIII). The writer estimates very roughly a computation-time of from three to four hours for a single computer when there are three aerial points to locate. With suitable division of the work, three computers should accomplish it in about one to two hours. These times are for experienced computers; mediocre ones or beginners would take longer. In connection with these estimates, the standard plates are supposed already to have been reduced.

T. E. Sterne

T. E. Sterne,
Captain, Ordnance Dept.